

# PHYSICS 523, GENERAL RELATIVITY

## Homework 2

Due Monday, 9<sup>th</sup> October 2006

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### Problem 1

Let frame  $\bar{\mathcal{O}}$  move with speed  $v$  in the  $x$ -direction relative to frame  $\mathcal{O}$ . A photon with frequency  $\nu$  measured in  $\mathcal{O}$  moves at an angle  $\theta$  relative to the  $x$ -axis.

a) We are to determine the frequency of the photon in  $\bar{\mathcal{O}}$ 's frame.

From the set up we know that the momentum of the photon in  $\mathcal{O}$  is<sup>1</sup>  $(E, E \cos \theta, E \sin \theta)$ —that this momentum is null is manifest. The energy of the photon is of course  $E = h\nu$  where  $h$  is Planck's constant and  $\nu$  is the frequency in  $\mathcal{O}$ 's frame.

Using the canonical Lorentz boost equation, the energy measured in frame  $\bar{\mathcal{O}}$  is given by

$$\begin{aligned}\bar{E} &= E\gamma - E \cos \theta v\gamma, \\ &= h\nu\gamma - h\nu \cos \theta v\gamma.\end{aligned}$$

But  $\bar{E} = h\bar{\nu}$ , so we see

$$\boxed{\therefore \frac{\bar{\nu}}{\nu} = \gamma(1 - v \cos \theta).} \tag{a.1}$$

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b) We are to find the angle  $\theta$  at which there is no Doppler shift observed.

All we need to do is find when  $\bar{\nu}/\nu = 1 = \gamma(1 - v \cos \theta)$ . Every five-year-old should be able to invert this to find that the angle at which no Doppler shift is observed is given by

$$\boxed{\therefore \cos \theta = \frac{1}{v} \left(1 - \sqrt{1 - v^2}\right).} \tag{b.1}$$

Notice that this implies that an observer moving close to the speed of light relative to the cosmic microwave background<sup>2</sup> will see a narrow 'tunnel' ahead of highly blue-shifted photons and large red-shifting outside this tunnel. As the relative velocity increases, the 'tunnel' of blue-shifted photons gets narrower and narrower.

c) We are asked to compute the result in part **a** above using the technique used above.

This was completed already. We made use of Schutz's equation (2.35) when we wrote the four-momentum of the photon in a manifestly light-like form, and we made use of Schutz's equation (2.38) when used the fact that  $E = h\nu$ .

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<sup>1</sup>We have aligned the axes so that the photon is travelling in the  $xy$ -plane. This is clearly a choice we are free to make.

<sup>2</sup>The rest frame of the CMB is defined to be that for which the CMB is mostly isotropic—specifically, the relative velocity at which no dipole mode is observed in the CMB power spectrum.

## Problem 2

Consider a very high energy cosmic ray proton, with energy  $10^9 m_p = 10^{18}$  eV as measured in the Sun's rest frame, scattering off of a cosmic microwave background photon with energy  $2 \times 10^{-4}$  eV. We are to use the Compton scattering formula to determine the maximum energy of the scattered photon.

We can guide our analysis with some simple heuristics. First of all, we are going to be interested in high momentum transfer interactions. In the proton rest frame we know from e.g. the Compton scattering formula that the hardest type of scattering occurs when the photon is fully 'reflected' with a scattering angle of  $\theta = \pi$ ; this is also what we would expect from classical physics<sup>3</sup>.

Now, imagine the proton travelling toward an observer at rest in the solar frame; any photons that scatter off the proton—ignoring their origin for the moment—will be blue-shifted (enormously) like a star would be, but only in the very forward direction of the proton. This means that the most energetic photons seen by an observer in the solar rest frame will be coming from those 'hard scatters' for which the final state photon travels parallel to the proton. Combining these two observations, we expect the most energetic scattering process will be that for which the photon and proton collide 'head-on' in the proton rest frame such that the momentum direction of the incoming photon is opposite to the incoming momentum of the proton in the solar frame.

We are now ready to verify this intuition and compute the maximum energy of the scattered photon. Before we start, it will be helpful to clear up some notation. We will work by translating between the two relevant frames in the problem, the proton rest frame and the solar rest frame. We may without loss of generality suppose that the proton is travelling in the positive  $x$ -direction with velocity  $v$ —with  $\gamma = (1 - v^2)^{-1/2}$ —in the solar frame. Also in the solar frame, we suppose there is some photon with energy  $E_\gamma^i = 2 \times 10^{-4}$  eV. This is the photon which we suppose to scatter off the proton.

The incoming photon's energy in the proton's rest frame we will denote  $\overline{E}_\gamma^i$ ; *in the proton frame*, we say that the angle between the photon's momentum and the positive  $x$ -axis is  $\overline{\theta}$ . After the photon scatters, it will be travelling at an angle  $\overline{\theta} - \overline{\varphi}$  relative to the  $x$ -axis, where  $\overline{\varphi}$  is the angle between the incoming and outgoing photon in the proton's rest frame. This outgoing photon will have energy denoted  $\overline{E}_\gamma^f$ . We can then boost this momentum back to the solar rest frame where its energy will be denoted  $E_\gamma^f$ .

From our work in problem 1 above, we know how to transform the energy of a photon between two frames with relative motion not parallel to the photon's direction. Let us begin our analysis by considering a photon in the proton's rest frame and determine what energy that photon had in the solar rest frame. Boosting along the  $(-x)$ -direction from the proton frame, we see that

$$E_\gamma^i = \overline{E}_\gamma^i \gamma (1 + v \cos \overline{\theta}) \quad \implies \quad \overline{E}_\gamma^i = \frac{E_\gamma^i}{\gamma (1 + v \cos \overline{\theta})}. \quad (\text{a.1})$$

We can relate the energy and scattering angle of the final-state photon in the proton rest frame using the Compton formula. Indeed, we see that

$$\overline{E}_\gamma^f = \frac{\overline{E}_\gamma^i m_p}{m_p + \overline{E}_\gamma^i (1 - \cos \overline{\varphi})}, \quad (\text{a.2})$$

where  $\overline{\varphi}$  is the scattering angle in this frame.

Finally, we need to reverse-boost the outgoing photon from the proton frame to the solar frame. Here, it is necessary to note that the relative angle between the outgoing photon and the  $x$ -axis is now  $\overline{\theta} - \overline{\varphi}$ . Recalling that we are boosting in the  $(-x)$ -direction again,

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<sup>3</sup>In the proton rest frame, however, this process does not look like what we're after: this process minimizes the out-state photon's energy in that frame. Nevertheless, it is the hardest type of scattering available—any other collision transfers less momentum between the proton and the photon.

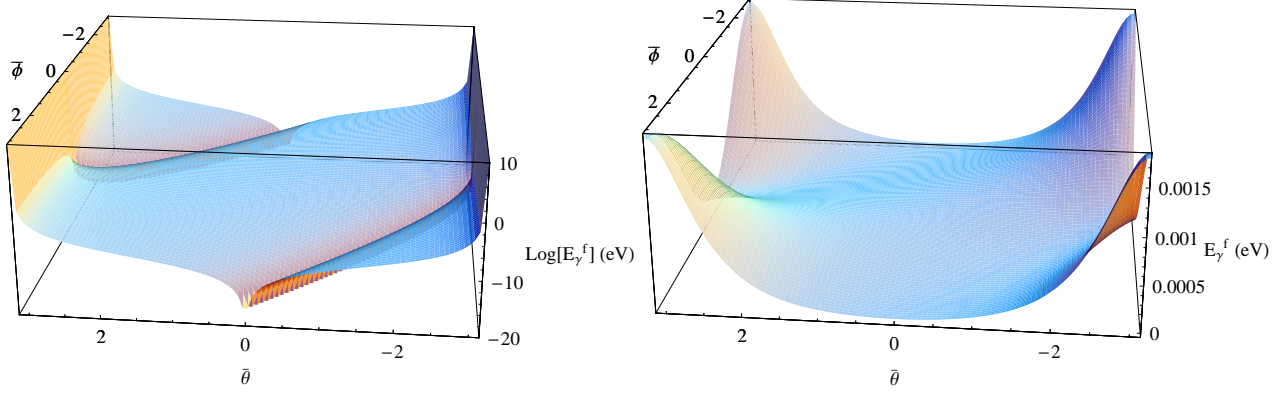


FIGURE 1. These graphs indicate the scattering energy of an incoming  $2 \times 10^{-4}$  eV photon, as measured in the solar frame, as a function of the incoming and outgoing angles  $\bar{\theta}$  and  $\bar{\varphi}$  as measured in the proton rest frame. The plot on the left is for the situation presently under investigation, where the proton energy is  $10^9 m_p$ ; because of the extremely wide-range of out-going photon energies, this is plotted on a log-scale. On the right is a simpler example where the cosmic ray proton is travelling only semi-relativistically with velocity  $v = 4c/5$ . In both cases it is clear that the maximal energy observed for scattering takes place when  $\bar{\varphi} = \bar{\theta} = \pi$ .

we see

$$E_\gamma^f = \bar{E}_\gamma^f \gamma \{1 + v \cos(\bar{\theta} - \bar{\varphi})\}. \quad (\text{a.3})$$

Putting all these together, we see that

$$\begin{aligned} E_\gamma^f &= \bar{E}_\gamma^f \gamma \{1 + v \cos(\bar{\theta} - \bar{\varphi})\}, \\ &= \frac{\bar{E}_\gamma^i m_p}{m_p + \bar{E}_\gamma^i (1 - \cos \bar{\varphi})} \gamma \{1 + v \cos(\bar{\theta} - \bar{\varphi})\}, \\ &= \frac{E_\gamma^i m_p}{m_p + E_\gamma^i \frac{(1 - \cos \bar{\varphi})}{\gamma(1 + v \cos \bar{\theta})}} \frac{\gamma \{1 + v \cos(\bar{\theta} - \bar{\varphi})\}}{\gamma (1 + v \cos \bar{\theta})}, \\ &\boxed{\therefore E_\gamma^f = \frac{E_\gamma^i m_p \{1 + v \cos(\bar{\theta} - \bar{\varphi})\}}{m_p (1 + v \cos \bar{\theta}) + E_\gamma^i \frac{1 - \cos \bar{\varphi}}{\gamma}}}. \end{aligned} \quad (\text{a.4})$$

The function above is plotted in Figure 1 along with the analogous result for a less-energetic cosmic ray proton.

At any rate, it is clear from the plot or a simple analysis of the second derivatives of  $E_\gamma^f$  that the global maximum is precisely at  $\bar{\theta} = \bar{\varphi} = \pi$ . This is exactly what we had anticipated—when the collision is head-on and the photon is scattered at an angle  $\pi$ . We can use this to strongly simplify the above equation (a.4),

$$\max \{E_\gamma^f\} = \frac{E_\gamma^i m_p (1 + v)}{m_p (1 - v) + 2E_\gamma^i \sqrt{1 - v^2}}. \quad (\text{a.5})$$

To actually compute the maximum energy allowed, we will need to put in numbers. We know that the energy of the proton is  $10^9 m_p = \gamma m_p$  so  $\gamma = 10^9$ . This is easily translated into a velocity of approximately  $1 - 5 \times 10^{-19}$ . Knowing that the mass of a proton is roughly  $10^9$  eV, the first term in the denominator of equation (a.5) is  $\mathcal{O} \sim 10^{-10}$  whereas the second term is  $\mathcal{O} \sim \times 10^{-13}$ , so to about a 1 percent accuracy (which is better than our proton mass figure anyway), we can approximate equation (a.5) as

$$\max \{E_\gamma^f\} \simeq E_\gamma^i \frac{1 + v}{1 - v} \approx \frac{2E_\gamma^i}{1 - v} = 8 \times 10^{14} \text{ eV}. \quad (\text{a.6})$$

Therefore, the maximum energy of a scattered CMB photon from a  $10^{18}$  eV cosmic ray proton is about 400 TeV—much higher than collider-scale physics. However, the rate of these types of hard-scatters is *enormously* low. Indeed, recalling the picture of a narrowing tunnel of blue-shift at high boost, we can use our work from problem 1 to see that only photons within a  $0.0025^\circ$  cone about the direction of motion of the proton are blue-shifted at all—and these are the only ones that can gain any meaningful energy from the collision. This amounts to a phase-space suppression of around  $10^{-10}$  even before we start looking at the small rate and low densities involved.

### Problem 3

Consider the coordinates  $u = t - x$  and  $v = t + x$  in Minkowski spacetime.

a) We are to define a  $u, v, y, z$ -coordinate system with the origin located at  $\{u = 0, v = 0, y = 0, z = 0\}$  with the basis vector  $\vec{e}_u$  connecting between the origin and the point  $\{u = 1, v = 0, y = 0, z = 0\}$  and similarly for  $\vec{e}_v$ . We are to relate these basis vectors to those in the normal Minkowski frame, and draw them on a spacetime plot in  $t, x$ -coordinates.

We can easily invert the defining equations  $u = t - x$  and  $v = t + x$  to find

$$t = \frac{u + v}{2} \quad \text{and} \quad x = \frac{v - u}{2}. \quad (\text{a.1})$$

Therefore, the the origin in  $u, v$ -coordinates is also the origin in  $tx$ -space. Also, the point where  $u = 1, v = 0$  which defines  $\vec{e}_u$  has coordinates  $t = \frac{1}{2}, x = -\frac{1}{2}$  in  $tx$ -space; the point  $u = 0, v = 1$  corresponds to  $t = \frac{1}{2}, x = \frac{1}{2}$  so that

$$\boxed{\vec{e}_u = \frac{\vec{e}_t - \vec{e}_x}{2} \quad \text{and} \quad \vec{e}_v = \frac{\vec{e}_t + \vec{e}_x}{2}.} \quad (\text{a.2})$$

These basis vectors are labeled on Figure 2.

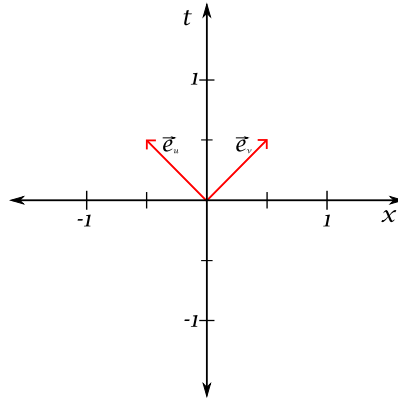


FIGURE 2. Figure required for problem 3 which shows the vectors  $\vec{e}_u$  and  $\vec{e}_v$  on the  $tx$ -plane.

b) We are to show that  $\{\vec{e}_u, \vec{e}_v, \vec{e}_y, \vec{e}_z\}$  span all of Minkowski space.

Because the map (a.2) is a bijection, the linear independence of  $\vec{e}_t$  and  $\vec{e}_x$  implies linear independence of  $\vec{e}_u$  and  $\vec{e}_v$ . And because these are manifestly linearly independent of  $\vec{e}_y$  and  $\vec{e}_z$ , the four vectors combine to form a linearly-independent set—which is to say that they span all of space.

c) We are to find the components of the metric tensor in this basis.

The components of the metric tensor in any basis  $\{\vec{e}_i\}$  is given by the matrix  $\tilde{g}_{ij} = g(\vec{e}_i, \vec{e}_j)$  where  $g(\cdot, \cdot)$  is the metric on spacetime. Because we have equation (a.2) which relates  $\vec{e}_u$  and  $\vec{e}_v$  to the  $tx$ -bases, we can compute all the relevant inner products using the canonical Minkowski metric. Indeed we find,

$$\tilde{g}_{ij} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{c.1})$$

d) We are to show that  $\vec{e}_u$  and  $\vec{e}_v$  are null but they are not orthogonal.

In part c above we needed to compute the inner products of all the basis vectors, including  $\vec{e}_u$  and  $\vec{e}_v$ . There we found that  $g(\vec{e}_u, \vec{e}_v) = 1/2$ , so  $\vec{e}_u$  and  $\vec{e}_v$  are not orthogonal. However,  $g(\vec{e}_u, \vec{e}_u) = g(\vec{e}_v, \vec{e}_v) = 0$ , so they are both null.

e) We are to compute the one-forms  $\mathbf{d}u$ ,  $\mathbf{d}v$ ,  $g(\vec{e}_u, \cdot)$ , and  $g(\vec{e}_v, \cdot)$ .

As scalar functions on spacetime, it is easy to compute the exterior derivatives of  $u$  and  $v$ . Indeed, using their respective definitions, we find immediately that

$$\mathbf{d}u = dt - dx \quad \text{and} \quad \mathbf{d}v = dt + dx. \quad (\text{e.1})$$

The only difference that arises when computing  $g(\vec{e}_u, \cdot)$ , for example, is that the *components* of  $\vec{e}_u$  are given in terms of the basis vectors  $\vec{e}_t$  and  $\vec{e}_x$  as in equation (a.2). Therefore in the usual Minkowski component notation, we have  $\vec{e}_u = (\frac{1}{2}, -\frac{1}{2}, 0, 0)$  and  $\vec{e}_v = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ . Using our standard Minkowski metric we see that

$$g(\vec{e}_u, \cdot) = -\frac{1}{2}dt - \frac{1}{2}dx \quad \text{and} \quad g(\vec{e}_v, \cdot) = -\frac{1}{2}dt + \frac{1}{2}dx. \quad (\text{e.2})$$

#### Problem 4

We are to give an example of four linearly independent null vectors in Minkowski space and show why it is not possible to make them all mutually orthogonal.

An easy example that comes to mind uses the coordinates  $\{x_-, x_+, y_+, z_+\}$  given by

$$x_- = t - x \quad x_+ = t + x \quad y_+ = t + y \quad z_+ = t + z. \quad (\text{a.1})$$

In case it is desirable to be condescendingly specific, this corresponds to taking basis vectors  $\{\vec{e}_{x_-}, \vec{e}_{x_+}, \vec{e}_{y_+}, \vec{e}_{z_+}\}$  where

$$\vec{e}_{x_-} = \frac{\vec{e}_t - \vec{e}_x}{2} \quad \vec{e}_{x_+} = \frac{\vec{e}_t + \vec{e}_x}{2} \quad \vec{e}_{y_+} = \frac{\vec{e}_t + \vec{e}_y}{2} \quad \vec{e}_{z_+} = \frac{\vec{e}_t + \vec{e}_z}{2}. \quad (\text{a.2})$$

It is quite obvious that each of these vectors is null, and because they are related to the original basis by an invertible map they still span the space. Again, to be specific<sup>4</sup>, notice that  $\vec{e}_t = \vec{e}_{x_-} + \vec{e}_{x_+}$  and so we may invert the other expressions by  $\vec{e}_i = 2\vec{e}_{i_+} - \vec{e}_t$  where  $i = x, y, z$ .

Let us now show that four linearly independent, null vectors cannot be simultaneously mutually orthogonal. We proceed via *reductio ad absurdum*: suppose that the set  $\{\vec{v}_i\}_{i=1, \dots, 4}$  were such linearly independent, mutually orthogonal and null. Because they are linearly independent, they can be used to define a basis which has an associated metric, say  $\tilde{g}$ . Now, as a matrix the entries of  $\tilde{g}$  are given by  $\tilde{g}_{ij} = \tilde{g}(\vec{v}_i, \vec{v}_j)$ ; because all the vectors are assumed to be orthogonal and null, all the entries of  $\tilde{g}$  are zero.

<sup>4</sup>Do you, ye grader, actually care for me to be this annoyingly specific?

This means that it has zero positive eigenvalues and zero negative eigenvalues—which implies  $\text{signature}(\bar{g}) = 0$ . But the signature of Minkowski spacetime must be  $\pm 2$ <sup>5</sup>, and this is basis-independent. —x—

To go one step further, the above argument actually implies that no null vector can be simultaneously orthogonal to and linearly independent of any three vectors.

### Problem 5

The frame  $\bar{\mathcal{O}}$  moves relative to  $\mathcal{O}$  with speed  $v$  in the  $z$ -direction.

a) We are to use the fact that the Abelian gauge theory field strength  $F_{\mu\nu}$  is a tensor to express the electric and magnetic field components measured in  $\bar{\mathcal{O}}$  in terms of the components measured in  $\mathcal{O}$ .

To determine the components of the field strength measured in frame  $\bar{\mathcal{O}}$  in terms of the components of frame  $\mathcal{O}$ , all we need to do is apply a Lorentz transformation for each of the two indices in  $F_{ab}$ :

$$F_{\bar{a}\bar{b}} = \Lambda_{\bar{a}}^c \Lambda_{\bar{b}}^d F_{cd}. \quad (\text{a.1})$$

Using some of our work in class to identify the components of  $F_{cd}$ , we may write the above expression in matrix notation<sup>6</sup> as

$$\begin{aligned} F_{\bar{a}\bar{b}} &= \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix}, \\ &= \begin{pmatrix} 0 & \gamma(E_x - vB_y) & \gamma(E_y + vB_x) & \gamma^2 E_z(1 - v^2) \\ -\gamma(E_x - vB_y) & 0 & -B_z & -\gamma(vE_x - B_y) \\ -\gamma(E_y + vB_x) & B_z & 0 & -\gamma(vE_y + B_x) \\ \gamma^2 E_z(v^2 - 1) & \gamma(vE_x - B_y) & \gamma(vE_y + B_x) & 0 \end{pmatrix}. \end{aligned}$$

Using the fact that  $\gamma^2(1 - v^2) = 1$ , we see that these imply

$$\bar{E}_x = \gamma(E_x - vB_y) \quad \bar{E}_y = \gamma(E_y + vB_x) \quad \bar{E}_z = E_z, \quad (\text{a.2})$$

$$\bar{B}_x = \gamma(B_x + vE_y) \quad \bar{B}_y = \gamma(B_y - vE_x) \quad \bar{B}_z = B_z. \quad (\text{a.3})$$

b) Say a particle of mass  $m$  and charge  $q$  is subjected to some electromagnetic fields. The particle is initially at rest in  $\bar{\mathcal{O}}$ 's frame. We are to calculate the components of its four-acceleration as measured in  $\bar{\mathcal{O}}$  at that moment, transform these components into those measured in  $\mathcal{O}$  and compare them with the equation for the particle's acceleration directly in  $\mathcal{O}$ 's frame.

We will use the fact that the four-acceleration is given by

$$\frac{dU^a}{d\tau} = \frac{q}{m} F^a_b U^b, \quad (\text{a.1})$$

where  $U^a$  is the four-velocity and  $\tau$  is some affine parameter along the particle's worldline. Now, the above equation works in any reference frame—we can substitute indices with bars over them if we'd like. Because the particle is initially at rest in frame  $\bar{\mathcal{O}}$ , its four velocity is given by  $U^{\bar{a}} = (-1, 0, 0, 0)$ . Therefore we can easily compute the

<sup>5</sup>The ' $\pm$ ' depends on convention. Actually, if you use complexified space *or* complexified time (which is more common, but still unusual these days), then you could get away with signature  $\pm 4$ .

<sup>6</sup>Here, as everywhere in every situation similar to this,  $\gamma = (1 - v^2)^{-1/2}$  where  $v$  is the velocity in question; in this case  $v$ .

four-acceleration in frame  $\bar{\mathcal{O}}$  as follows:

$$\begin{aligned} \frac{dU^{\bar{a}}}{d\tau} &= \frac{q}{m} F^{\bar{a}}_{\bar{b}} U^{\bar{b}} = \frac{q}{m} g^{\bar{a}\rho} F_{\rho\bar{b}} U^{\bar{b}}, \\ &= \frac{q}{m} \begin{pmatrix} 0 & -\bar{E}_x & -\bar{E}_y & -\bar{E}_z \\ -\bar{E}_x & 0 & -\bar{B}_z & \bar{B}_y \\ -\bar{E}_y & \bar{B}_z & 0 & -\bar{B}_x \\ -\bar{E}_z & -\bar{B}_y & \bar{B}_x & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ &= \frac{q}{m} (0, \bar{E}_x, \bar{E}_y, \bar{E}_z). \end{aligned}$$

This results shows us that sometimes relativity is terribly unnecessary—the result is completely obvious from a classical electrodynamics point of view.

To determine the components of the four-acceleration as viewed in frame  $\mathcal{O}$ , all we need to do is Lorentz transform the components of the four-acceleration back into  $\mathcal{O}$  (because it is a vector). We find then that the four-acceleration in  $\mathcal{O}$  is given by

$$\begin{aligned} \frac{dU^a}{d\tau} &= \Lambda^a_{\bar{a}} \frac{dU^{\bar{a}}}{d\tau} = \frac{q}{m} \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ \bar{E}_x \\ \bar{E}_y \\ \bar{E}_z \end{pmatrix}, \\ &= \frac{q}{m} \begin{pmatrix} v\gamma\bar{E}_z \\ \bar{E}_x \\ \bar{E}_y \\ \gamma\bar{E}_z \end{pmatrix}, \end{aligned}$$

which upon substitution of the  $\bar{\mathcal{O}}$  field components in terms of the  $\mathcal{O}$  components, implies

$$\frac{dU^a}{d\tau} = \frac{q\gamma}{m} \begin{pmatrix} vE_z \\ E_x - vB_y \\ E_y + vB_x \\ E_z \end{pmatrix}. \quad (\text{a.2})$$

Now, to compute this directly in frame  $\mathcal{O}$ , we need only transform the four-velocity vector  $U^{\bar{a}}$  into  $U^a$ ,

$$U^a = \Lambda^a_{\bar{a}} U^{\bar{a}} = \begin{pmatrix} -\gamma \\ 0 \\ 0 \\ -v\gamma \end{pmatrix}, \quad (\text{a.3})$$

and use this in the expression for the four-acceleration for an Abelian field theory as quoted above. So we have

$$\frac{dU^a}{d\tau} = \frac{q}{m} F^a_b U^b = \frac{q}{m} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \cdot \begin{pmatrix} -\gamma \\ 0 \\ 0 \\ -v\gamma \end{pmatrix}, \quad (\text{a.4})$$

$$\therefore \frac{dU^a}{d\tau} = \frac{q\gamma}{m} \begin{pmatrix} vE_z \\ E_x - vB_y \\ E_y + vB_x \\ E_z \end{pmatrix}. \quad (\text{a.5})$$